Homogeneous phase spaces: the Cayley–Klein framework

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Abstract

The metric structure of homogeneous spaces of rank-one and rank-two associated to the real pseudo-orthogonal groups $SO(p,q)$ and some of their contractions (e.g., $ISO(p,q)$, Newton–Hooke type groups…) is studied. All these spaces are described from a unified setting following a Cayley–Klein scheme allowing to simultaneously study the main features of their Riemannian, pesudoRiemannian and semiRiemannian metrics, as well as of their curvatures. Some of the rank-one spaces are naturally interpreted as spacetime models. Likewise, the same natural interpretation for rank-two spaces is as spaces of lines in rank-one spaces; through this relation these rank-two spaces give rise to homogeneous phase space models. The main features of the phase spaces for homogeneous spacetimes are analysed.

1 Introduction

The main aspect usually considered when working on a phase space is its associated symplectic structure. However *homogeneous* phase spaces, i.e., those admitting a structure preserving a Lie group of transformations have a richer structure, which can be overlooked if attention is focused only on their symplectic structure. Actually, homogeneous phase spaces have also a canonical connection and a metric structure, with a 'main' Riemannian metric (which can be as well pseudoRiemannian or degenerate Riemannian). In some cases, the phase spaces have also invariant foliations, with a subsidiary metric defined in each leaf.

The aim of this paper is to provide a complete characterization of the quadratic metric in a set of phase spaces which are constructed as symmetrical homogeneous spaces coming from the orthogonal Cayley–Klein (CK) groups [1, 2]. This family of real Lie groups, called 'quasi-orthogonal' [3] are exactly the family of motion groups of the geometries of a real space with a projective metric [4, 5]. They include the semisimple pseudo-orthogonal groups of the Cartan series B_l, D_l as well as many others which are non-semisimple and that can be obtained by contraction processes from the formers (e.g., Euclidean, Poincaré, Galilean, Newton–Hooke type groups...). All groups in this family share many important properties allowing their study to be done at once. Furthermore, the kinematical groups associated to different homogeneous models of spacetime [6] belong to this family, and this fact indeed provides one of the strongest physical motivations to study them.

We first introduce in Section 2 the family of orthogonal CK groups from an 'abstract' point of view. In Section 3 we focus on the set of rank-one homogeneous spaces associated to this family, and we describe in detail how the metric(s) in these spaces comes from the Killing–Cartan form. Physically, all homogeneous models of spacetime are rank-one spaces. In Section 4 we carry out a similar study for the rank-two spaces; this is also physically meaningful, because homogeneous phase spaces are rank-two spaces. The last Section is devoted to commenting upon the physical meaning of properties of the homogeneous phase spaces, which are presented against the bakground provided by the more familiar properties of spacetime models.

2 Symmetrical homogeneous CK spaces

The orthogonal CK algebras are real Lie algebras of dimension $N(N+1)/2$ whose generators are Ω_{ab} with $a, b = 0, 1, \ldots, N$ and $a < b$. This family can be described collectively by means of N real coefficients $\omega_1, \ldots, \omega_N$. The non-zero Lie brackets are (no sum over repeated indices):

$$[\Omega_{ab}, \Omega_{ac}] = \omega_{ab}\Omega_{bc} \qquad [\Omega_{ab}, \Omega_{bc}] = -\Omega_{ac} \qquad [\Omega_{ac}, \Omega_{bc}] = \omega_{bc}\Omega_{ab} \tag{2.1}$$

with $a < b < c$, and where the coefficients with two indices are defined by

$$\omega_{ab} := \omega_{a+1}\omega_{a+2}\cdots\omega_b \qquad a, b = 0, 1, \ldots, N \qquad a < b \tag{2.2}$$

satisfying

$$\omega_{ac} = \omega_{ab}\omega_{bc} \qquad \omega_a = \omega_{a-1\,a}. \tag{2.3}$$

We will denote $\mathfrak{so}_{\omega_1,\ldots,\omega_N}(N+1)$ the generic algebra in the orthogonal CK family [1, 2]. These algebras have a fundamental or *vector* representation by $(N+1)\times(N+1)$ real matrices

$$\Omega_{ab} \to -\omega_{ab}e_{ab} + e_{ba} \tag{2.4}$$

where e_{ab} is the matrix with a single non-zero entry, 1, in the row a and column b. By exponentiation this representation allows to define the orthogonal CK groups

denoted $SO_{\omega_1,\dots,\omega_N}(N+1)$, whose one-parameter subgroups are easy to compute:

$$e^{x\Omega_{ab}} = \sum_{\substack{s=0\\ s\neq a,b}}^{N} e_{ss} + \mathrm{C}_{\omega_{ab}}(x)(e_{aa}+e_{bb}) + \mathrm{S}_{\omega_{ab}}(x)(-\omega_{ab}e_{ab}+e_{ba}) \tag{2.5}$$

where we introduce the 'labeled' cosine $\mathrm{C}_\omega(x)$ and sine $\mathrm{S}_\omega(x)$ functions defined by [7]:

$$\mathrm{C}_\omega(x) = \begin{cases} \cos\sqrt{\omega}\,x & \omega>0\\ 1 & \omega=0\\ \cosh\sqrt{-\omega}\,x & \omega<0 \end{cases} \qquad \mathrm{S}_\omega(x) = \begin{cases} \frac{1}{\sqrt{\omega}}\sin\sqrt{\omega}\,x & \omega>0\\ x & \omega=0\\ \frac{1}{\sqrt{-\omega}}\sinh\sqrt{-\omega}\,x & \omega<0 \end{cases} \tag{2.6}$$

Each coefficient ω_a can be scaled to the values $+1$, 0, -1, thus the family $SO_{\omega_1,\dots,\omega_N}(N+1)$ essentially comprises 3^N Lie groups; we remark that a value of any coefficient ω_a equal to zero is equivalent to a contraction limit. The properties of groups in the CK family depend mainly on whether the values of the coefficients ω_a in the sequence $(\omega_1,\dots,\omega_N)$ are equal to zero or not. The essential properties are [1]:

• When all $\omega_a \neq 0\ \forall a$, the group $SO_{\omega_1,\dots,\omega_N}(N+1)$ is isomorphic to a semisimple pseudo-orthogonal group $SO(p,q)$ $(p+q=N+1)$ in the Cartan series B_l and D_l. Their vector matrix representation (2.4) acting in $\mathbb{R}^{N+1}$ via matrix multiplication leaves invariant a quadratic form whose matrix is $\Lambda_0^{(1)} = \mathrm{diag}\,(1,\omega_{01},\omega_{02},\dots,\omega_{0N})$, so the values (p,q) can be determined as the number of positive and negative terms in this sequence.

• When a constant $\omega_a=0$, the group $SO_{\omega_1,\dots,\omega_{a-1},\omega_a=0,\omega_{a+1},\dots,\omega_N}(N+1)$ has a semidirect structure:

$$SO_{\omega_1,\dots,\omega_{a-1},\omega_a=0,\omega_{a+1},\dots,\omega_N}(N+1) \equiv T \odot (SO_{\omega_1,\dots,\omega_{a-1}}(a) \otimes SO_{\omega_{a+1},\dots,\omega_N}(N+1-a))$$

where T is an abelian subgroup of dimension $a(N+1-a)$.

Repeated application of these results leads to an explicit description of the structure of CK groups according to the values of the constants ω_i:

• Only $\omega_1=0$, other different from zero. We find the inhomogeneous groups with a semidirect product structure:

$$SO_{0,\omega_2,\dots,\omega_N}(N+1) \equiv T_N \odot SO_{\omega_2,\dots,\omega_N}(N) \equiv ISO(p,q) \qquad p+q=N.$$

The Abelian subgroup is T_N generated by $\langle \Omega_{0b};\ b=1,\dots,N\rangle$ and $SO_{\omega_2,\dots,\omega_N}(N)$ is a pseudo-orthogonal group which preserves a quadratic form whose matrix is $\mathrm{diag}\,(+,\omega_{12},\dots,\omega_{1N})$. The Euclidean group $ISO(N)$ appears in this case when $(\omega_1,\omega_2,\dots,\omega_N)=(0,+,\dots,+)$; the Poincaré group $ISO(N-1,1)$ is reproduced several times, e.g. for $(0,-,+,\dots,+)$, $(0,+,\dots,+,-)$, etc.

• $\omega_1=\omega_2=0$, other different from zero. Here we have two different semidirect structures for the CK group. The one associated with the vanishing of ω_1 is:

$$SO_{0,0,\omega_3,\dots,\omega_N}(N+1) \equiv T_N \odot SO_{0,\omega_3,\dots,\omega_N}(N)$$

where the second factor has again a semidirect structure due to the vanishing of ω_2:

$$SO_{0,0,\omega_3,\dots,\omega_N}(N+1) \equiv T_N \odot (T_{N-1} \odot SO_{\omega_3,\dots,\omega_N}(N-1)) \equiv IISO(p,q)$$

with $p+q=N-1$. The alternative semidirect structure can be written similarly. The Galilean group $IISO(N-1)$ appears in this case associated to $(0,0,+,\dots,+)$.

• $\omega_a=0$, $a \notin \{1,N\}$. These groups have a structure $T_{a(N+1-a)} \odot (SO(p,q) \otimes SO(p',q'))$ [8]. In particular, for $\omega_2=0$ we have $T_{2N-2} \odot (SO(p,q) \otimes SO(p',q'))$ with $p+q=N-1$ and $p'+q'=2$, which include for $q=0$ the oscillating and expanding Newton–Hooke groups [6] associated to $(+,0,+,\dots,+)$ and $(-,0,+,\dots,+)$, respectively.

• The extreme contracted case in the CK family corresponds to setting all constants $\omega_a=0$. This is the so-called flag space group $SO_{0,\dots,0}(N+1) \equiv I\dots ISO(1)$ [3]. In our notation it should be understood $ISO(1) \equiv \mathbb{R}$.

The CK algebra $\mathfrak{so}_{\omega_1,\dots,\omega_N}(N+1)$ can be endowed with an Abelian group $\mathbb{Z}_2^{\otimes N}$ of involutive automorphisms generated by N involutions: $\Theta^{(1)},\dots,\Theta^{(N)}$. The action of $\Theta^{(m)}$ on the generators Ω_{ab} is as follows:

$$\Theta^{(m)}(\Omega_{ab}) = \begin{cases} \Omega_{ab} & \text{if either } a \geq m \text{ or } b < m \\ -\Omega_{ab} & \text{if } a < m \text{ and } b \geq m \end{cases} \tag{2.7}$$

Each involution $\Theta^{(m)}$ provides a Cartan-like decomposition of the CK algebra in antiinvariant and invariant subspaces, denoted $\mathfrak{p}^{(m)}$ and $\mathfrak{h}^{(m)}$, respectively:

$$\mathfrak{so}_{\omega_1,\dots,\omega_N}(N+1) = \mathfrak{p}^{(m)} \oplus \mathfrak{h}^{(m)}. \tag{2.8}$$

The set $\mathfrak{h}^{(m)}$ of invariant elements is a Lie subalgebra, with a direct sum structure:

$$\mathfrak{h}^{(m)} = \mathfrak{so}_{\omega_1,\dots,\omega_{m-1}}(m) \oplus \mathfrak{so}_{\omega_{m+1},\dots,\omega_N}(N+1-m), \tag{2.9}$$

while the vector subspace $\mathfrak{p}^{(m)}$ is not always a subalgebra. The decomposition (2.8) can be graphically visualized by arranging the generators of $\mathfrak{so}_{\omega_1,\dots,\omega_N}(N+1)$ in the form of a triangle

$$\begin{array}{cccc|cccc}
\Omega_{01} & \Omega_{02} & \dots & \Omega_{0\,m-1} & \Omega_{0m} & \Omega_{0\,m+1} & \dots & \Omega_{0N} \\
 & \Omega_{12} & \dots & \Omega_{1\,m-1} & \Omega_{1m} & \Omega_{1\,m+1} & \dots & \Omega_{1N} \\
 & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
 & & & \Omega_{m-2\,m-1} & \Omega_{m-2\,m} & \Omega_{m-2\,m+1} & \dots & \Omega_{m-2\,N} \\
 & & & & \Omega_{m-1\,m} & \Omega_{m-1\,m+1} & \dots & \Omega_{m-1\,N} \\
\cline{5-8}
\multicolumn{4}{c}{} & & \Omega_{m\,m+1} & \dots & \Omega_{m\,N} \\
\multicolumn{4}{c}{} & & & \ddots & \vdots \\
\multicolumn{4}{c}{} & & & & \Omega_{N-1\,N}
\end{array}$$

The subspace $\mathfrak{p}^{(m)}$ is spanned by those $m(N+1-m)$ generators inside the rectangle; the left and down triangles correspond respectively to the subalgebras $\mathfrak{so}_{\omega_1,\dots,\omega_{m-1}}(m)$ and $\mathfrak{so}_{\omega_{m+1},\dots,\omega_N}(N+1-m)$ of $\mathfrak{h}^{(m)}$.

Relative to the decomposition (2.9), the structure of commutators (2.1) in the CK algebra can be summed up as:

$$[\mathfrak{h}^{(m)}, \mathfrak{h}^{(m)}] \subset \mathfrak{h}^{(m)} \qquad [\mathfrak{h}^{(m)}, \mathfrak{p}^{(m)}] \subset \mathfrak{p}^{(m)} \qquad [\mathfrak{p}^{(m)}, \mathfrak{p}^{(m)}] \subset \mathfrak{h}^{(m)}. \tag{2.10}$$

In the special case $\omega_m = 0$ the last equation reduces to $[\mathfrak{p}^{(m)}, \mathfrak{p}^{(m)}] = 0$, and in this case $\mathfrak{p}^{(m)}$ is not only a subspace, but an an ideal.

All algebras in the family $\mathfrak{so}_{\omega_1,\ldots,\omega_N}(N+1)$ (no matter of how many values ω_i are equal to zero) have an asociated Killing–Cartan metric form which can be defined in the same way as when the algebra is simple, by the trace of the product of the adjoint representation of the generators:

$$g(\Omega_{ab}, \Omega_{cd}) = \mathrm{Trace}(\mathrm{ad}\Omega_{ab} \cdot \mathrm{ad}\Omega_{cd}). \tag{2.11}$$

A simple calculation shows that the basis Ω_{ab} diagonalises the Killing–Cartan form:

$$g(\Omega_{ab}, \Omega_{cd}) = -2(N-1)\delta_{ac}\delta_{bd}\omega_{ab}. \tag{2.12}$$

The Killing–Cartan form is only non-degenerate when the algebra is simple: in fact, only when all ω_i are different from zero the diagonal values $g(\Omega_{ab}, \Omega_{ab}) = -2(N-1)\omega_{ab}$ are all different from zero. As soon as a constant ω_i is made zero, this property is lost. When a *single* $\omega_a = 0$, then the subspace where the Killing–Cartan form vanishes is exactly $\mathfrak{p}^{(a)}$, while the restriction to the subalgebra $\mathfrak{h}^{(a)}$ is non-degenerate.

Each $\mathfrak{h}^{(m)}$ generates a subgroup $H^{(m)}$ of the CK group which provides a quotient space

$$\mathcal{S}^{(m)} \equiv SO_{\omega_1,\ldots,\omega_N}(N+1) \Big/ \Big(SO_{\omega_1,\ldots,\omega_{m-1}}(m) \otimes SO_{\omega_{m+1},\ldots,\omega_N}(N+1-m)\Big) \,. \tag{2.13}$$

The dimension of $\mathcal{S}^{(m)}$ is that of $\mathfrak{p}^{(m)}$; in fact, $\mathfrak{p}^{(m)}$ is canonically identified with the tangent space to $\mathcal{S}^{(m)}$ at the origin:

$$\dim(\mathcal{S}^{(m)}) = m(N+1-m). \tag{2.14}$$

Then $\mathcal{S}^{(m)}$ is a symmetrical homogeneous space, and will be generically called *orthogonal CK space*. There are exactly N such symmetrical homogeneous spaces $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$, ..., $\mathcal{S}^{(N)}$ associated to each CK group $SO_{\omega_1,\ldots,\omega_N}(N+1)$; for the moment we will understand the values $\omega_1, \ldots, \omega_N$ as already fixed when we consider the aggregate of spaces $\mathcal{S}^{(m)}$.

These N spaces, although different, are not completely unrelated, and it is possible to reformulate all properties of any given space in terms of any other space, say $\mathcal{S}^{(m)}$, in the aggregate. Although this possibility exists for any m, it is most easily understood when interpreting $\mathcal{S}^{(2)}, \ldots, \mathcal{S}^{(N)}$ in terms of $\mathcal{S}^{(1)}$. As a matter of fact, the spaces $\mathcal{S}^{(1)}$ associated to the CK algebras $so_{\omega_1,\ldots,\omega_N}(N+1)$ are rather well known (in particular when the constants $\omega_2, \omega_3, \ldots \omega_N$ are all positive they are the constant curvature Riemannian spaces, cf. next Section). Now the key for the interpretation of $\mathcal{S}^{(2)}, \ldots, \mathcal{S}^{(N)}$ in terms of $\mathcal{S}^{(1)}$ lies in the fact that the subgroups $H^{(m)}$

$(m = 1, 2, \ldots, N)$ are identified with the isotropy subgroups of a point $(m = 1)$, a line $(m = 2)$,..., a hyperplane $(m = N)$ in $\mathcal{S}^{(1)}$. Hence, $\mathcal{S}^{(1)}$ being taken as *the* space, its elements being called *points*, $\mathcal{S}^{(2)}$ is the space of all lines in $\mathcal{S}^{(1)}$, $\mathcal{S}^{(3)}$ is the space of all 2-planes in $\mathcal{S}^{(1)}$, and so on. This view allows a large freedom for translating properties of any of the spaces $\mathcal{S}^{(2)}, \mathcal{S}^{(3)}, \ldots,$ to properties of lines, 2-planes, ..., in $\mathcal{S}^{(1)}$. In some cases, this translation gives a much clearer picture than it would be possible by blindly working with each space $\mathcal{S}^{(m)}$.

An important feature of homogeneous spaces is their rank. When dealing with homogeneous spaces associated to simple Lie groups, the rank of the space (not to be confused with the rank of the group itself) is usually defined as the maximal dimension of a totally geodesic flat submanifold [9]. An alternative definition is preferable in the context of CK homogeneous spaces: we will define the *rank* of a CK homogeneous space $\mathcal{S}^{(m)}$ as the number of independent invariants under the action of the CK group for each generic pair of elements in the space $\mathcal{S}^{(m)}$. This number was first determined by Jordan [10] when the group is the motion group of the N-dimensional Euclidean space; it has a single invariant (the ordinary distance) associated to each pair of points, two invariants for each pair of lines (an angle and a distance between the two lines), and, in general, $\min(m, N+1-m)$ invariants for a pair of $(m-1)$-planes (these invariants are stationary angles and a single stationary distance). The reason why this definition of rank is better in the CK context is that the total number of invariant stationary angles and distances obtained by Jordan turns out to be the same for all spaces in the CK family, i.e., do not depend on the values ω_i:

$$\operatorname{rank}(\mathcal{S}^{(m)}) = \min(m, N+1-m). \tag{2.15}$$

In addition to the rank, which is an essential property making the spaces $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$, ..., $\mathcal{S}^{(N)}$ rather different among themselves, the fact that they are homogeneous spaces of the same Lie group entitles the space $\mathcal{S}^{(m)}$ to inherit from its Lie algebra/group the following geometrical structures [1]:

- A structure of symmetrical homogeneous space, which leads to a canonical connection invariant under the CK group.
- A (possibly degenerate) quadratic main metric, coming from a suitable rescaling of a Killing–Cartan form. The main metric in the space $\mathcal{S}^{(m)}$ is non-degenerate when the constants $\omega_1, \omega_2, \ldots, \omega_{m-1}, \omega_{m+1}, \ldots, \omega_N$ are all different from zero (note that ω_m is missing in this list); in this case its Levi–Civita connection coincides with the canonical connection.
- When one of the constants $\omega_1, \omega_2, \ldots, \omega_{m-1}, \omega_{m+1}, \ldots, \omega_N$ is equal to zero, then the main metric is degenerate and the space $\mathcal{S}^{(m)}$ has an invariant foliation. This can be considered as a fibered structure, each of whose leaves carries a subsidiary metric, coming again from the Killing–Cartan form through restriction to the leaves and suitable rescaling.
- Sectional curvatures of the space $\mathcal{S}^{(m)}$ are completely determined by the value ω_m. Only the rank-one spaces $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(N)}$ are of *constant curvature*; CK spaces of higher rank are not of constant curvature as this is usually understood, yet their structure is as close to constant curvature as a higher rank space can allow, because

higher rank spaces will have neccesarily to contain completely geodesic flat submanifolds of dimension at least equal to the rank (and exactly equal to the rank when the group is simple).

• Finally, the canonical connection and the complete hierarchy of subsidiary metrics are compatible.

3 Rank-one spaces

For each CK group $SO_{\omega_1,\ldots,\omega_N}(N+1)$, the CK space $\mathcal{S}^{(1)}$ is the rank-one symmetrical homogeneous space obtained as the quotient by the subgroup $H^{(1)}$:

$$\mathcal{S}^{(1)} \equiv SO_{\omega_1,\ldots,\omega_N}(N+1)\,/SO_{\omega_2,\ldots,\omega_N}(N)\,. \tag{3.1}$$

The dimension of the space $\mathcal{S}^{(1)}$ is N. The subgroup $H^{(1)}$ is generated by the subalgebra $\mathfrak{h}^{(1)}$ of the Cartan decomposition associated to the involution $\Theta^{(1)}$: $\mathfrak{so}_{\omega_1,\ldots,\omega_N}(N+1) = \mathfrak{p}^{(1)} \oplus \mathfrak{h}^{(1)}$. A good choice when dealing with a space of type $\mathcal{S}^{(m)}$ is to replace the notation Ω_{ab} by a new 'rank-adapted' notation which conveys the interpretation of generators either as translations or as rotations in $\mathcal{S}^{(m)}$. Here, for $\mathcal{S}^{(1)}$ we denote the generators Ω_{0i} in $\mathfrak{p}^{(1)}$ as P_i and those Ω_{ij} in $\mathfrak{h}^{(1)}$ as J_{ij} $(i,j=1,\ldots,N,\ i<j)$ according to the following arrangement:

$$\begin{array}{|ccccc} \Omega_{01} & \Omega_{02} & \Omega_{03} & \ldots & \Omega_{0N} \\ \hline & \Omega_{12} & \Omega_{13} & \ldots & \Omega_{1N} \\ & & \Omega_{23} & \ldots & \Omega_{2N} \\ & & & \ddots & \vdots \\ & & & & \Omega_{N-1\,N} \end{array} \qquad \equiv \qquad \begin{array}{|ccccc} P_1 & P_2 & P_3 & \ldots & P_N \\ \hline & J_{12} & J_{13} & \ldots & J_{1N} \\ & & J_{23} & \ldots & J_{2N} \\ & & & \ddots & \vdots \\ & & & & J_{N-1\,N} \end{array}$$

and we call now κ_i and κ_{ij} the coefficients ω_i, and ω_{ij}. In this notation the non-zero Lie brackets of the CK algebra (2.1) clearly display properties of translations and rotations in $\mathcal{S}^{(1)}$:

$$\begin{array}{lll} [J_{ij}, J_{ik}] = \kappa_{ij} J_{jk} & [J_{ij}, J_{jk}] = -J_{ik} & [J_{ik}, J_{jk}] = \kappa_{jk} J_{ij} \\ [J_{ij}, P_i] = P_j & [J_{ij}, P_j] = -\kappa_{ij} P_i & \\ [P_i, P_j] = \kappa_1 \kappa_{1i} J_{ij} & & \end{array} \tag{3.2}$$

with $i<j<k$, $i,j,k=1,\ldots,N$; a symbol like κ_{11} with two equal indices will be always understood as equal to 1. Notice that the constant κ_1 *only* appears in the commutators of translations, foreshadowing its role as the curvature of the space.

When the constant κ_1 is equal to zero, and only in this case, the CK group acts on the space as a group of linear-affine transformations. In other cases the group action is intrinsically non-linear. But it is possible to *linearize* the action for *all* CK spaces $\mathcal{S}^{(1)}$ by going to some ambient space. The tool to do this is the *vector* representation of $SO_{\omega_1,\ldots,\omega_N}(N+1)$; in this representation the generators are given by the matrices:

$$\Omega_{0i} \to P_i^{(1)} = -\kappa_{0i}\, e_{0i} + e_{i0} \qquad \Omega_{ij} \to J_{ij}^{(1)} = -\kappa_{ij}\, e_{ij} + e_{ji} \tag{3.3}$$

each of which satisfy the condition $X^T\Lambda_0^{(1)} + \Lambda_0^{(1)}X$, where $\Lambda_0^{(1)}$ is the matrix

$$\Lambda_0^{(1)} = \operatorname{diag}(1, \omega_{01}, \omega_{02}, \ldots, \omega_{0N}). \tag{3.4}$$

This representation of the Lie algebra produces the *vector* representation of the CK group $SO_{\omega_1,\ldots,\omega_N}(N+1)$ as a group of matrices of order $N+1$, which acts naturally and linearly (via matrix multiplication) in $\mathbb{R}^{N+1} = (x^0, x^1, \ldots, x^N)$. This action has two properties which are relevant for our purposes:

- It leaves invariant a quadratic form whose matrix is $\Lambda_0^{(1)}$
- The subgroup $H^{(1)}$ generated by the subalgebra $\mathfrak{h}^{(1)}$ is the isotopy subgroup of the point $O = (1, 0, \ldots, 0) \in \mathbb{R}^{N+1}$, i.e., the origin of $\mathcal{S}^{(1)}$.

Hence, the space $\mathcal{S}^{(1)}$ can be identified with the orbit of O under the linear action of the group $SO_{\omega_1,\ldots,\omega_N}(N+1)$ in the space $\mathbb{R}^{N+1}$. This action is by isometries of the metric $\Lambda_0^{(1)}$, so it cannot be transitive in $\mathbb{R}^{N+1}$, but only transitive on each orbit, which should be neccesarily contained in the 'sphere'

$$(x^0)^2 + \sum_{l=1}^{N} \kappa_{0l}(x^l)^2 = 1. \tag{3.5}$$

The $N+1$ coordinates $(x^0, x^1, \ldots, x^N)$ are called *Weierstrass coordinates* for the CK space $\mathcal{S}^{(1)}$; their importance stems from the linear character of the group action on them. There are two other natural coordinate systems in $\mathcal{S}^{(1)}$, which are of a type called *geodesic* in differential geometry:

- The point $\exp(a^1P_1)\exp(a^2P_2)\ldots\exp(a^NP_N)O$ has $(a^1, \ldots, a^N)$ as *geodesic parallel coordinates.*
- The point $\exp(\theta^N J_{N-1,N})\ldots\exp(\theta^2 J_{12})\exp(\theta^1 P_1)\,O$ has $(\theta^1, \ldots, \theta^N)$ as *geodesic polar coordinates.*

In particular, for the relationship between geodesic parallel and Weierstrass coordinates we get

$$x^0 = \prod_{l=1}^{N} \mathrm{C}_{\kappa_{0l}}(a^l) \qquad x^i = \mathrm{S}_{\kappa_{0i}}(a^i) \prod_{l=i+1}^{N} \mathrm{C}_{\kappa_{0l}}(a^l) \qquad x^N = \mathrm{S}_{\kappa_{0N}}(a^N). \tag{3.6}$$

When $\kappa_1 = 0$, the sphere (3.5) reduces to an affine hyperplane in $\mathbb{R}^{N+1}$ with equation $x^0 = 1$, and geodesic parallel coordinates are simply cartesian coordinates in this hyperplane.

There are two ways to characterize the metric structure of $\mathcal{S}^{(1)}$. The intrinsic one starts by translating the Killing–Cartan form (2.12) to the rank-one language. The diagonal non-zero values are: $g(P_i, P_i) = -2(N-1)\kappa_1\kappa_{1i}$, $g(J_{jk}, J_{jk}) = -2(N-1)\kappa_{jk}$ so the restriction of the Killing–Cartan form to the subspace $\mathfrak{p}^{(1)}$ can be written as:

$$g\,|_{\mathfrak{p}^{(1)}} = -2(N-1)\kappa_1 g^{(1)} \tag{3.7}$$

where $g^{(1)}$, the natural candidate for the metric in the tangent space $\mathfrak{p}^{(1)}$, is:

$$g^{(1)}(P_i, P_j) = \delta_{ij}\kappa_{1i}. \tag{3.8}$$

The matrix of the metric $g^{(1)}$ in the tangent space $\mathfrak{p}^{(1)}$ is, in the canonical basis $(P_1, P_2, \ldots, P_N)$:

$$\Lambda^{(1)} = \operatorname{diag}(+, \kappa_{12}, \kappa_{13}, \ldots, \kappa_{1N}) = \operatorname{diag}(+, \kappa_2, \kappa_2\kappa_3, \ldots, \kappa_2 \cdots \kappa_N). \tag{3.9}$$

Thus the constants $\kappa_2, \kappa_3, \ldots, \kappa_N$ determine the signature of the main metric at the origin in the space $\mathcal{S}^{(1)}$. This metric can be translated to all points in the CK space $\mathcal{S}^{(1)}$ by the group action, so that the action is by isometries. Now the curvature of this metric can be computed and turns out to be constant and equal to κ_1. This completes the geometric interpretation of the constants κ_i in the geometry of the space $\mathcal{S}^{(1)}$, and suggest a more detailed notation for these rank-one spaces: $\mathcal{S}^{(1)} \equiv \mathcal{S}^{[\kappa_1]\kappa_2,\ldots,\kappa_N}$.

When all the constants $\kappa_2, \kappa_3, \ldots, \kappa_N$ are different from zero, then the metric is non-degenerate (Riemannian or pseudoRiemannian case), and it is definite positive when all of them are positive (Riemannian case). Otherwise, when a given $\kappa_a = 0$, $(a = 2, \ldots, N)$, the metric is degenerate and we introduce the following decomposition for $\mathfrak{p}^{(1)}$:

$$\mathfrak{p}^{(1)} = \mathfrak{b}_a{}^{(1)} \oplus \mathfrak{f}_a^{(1)} \qquad \mathfrak{b}_a^{(1)} = \langle P_1, \ldots, P_{a-1} \rangle \qquad \mathfrak{f}_a^{(1)} = \langle P_a, \ldots, P_N \rangle. \tag{3.10}$$

In this case $\mathfrak{f}_a^{(1)}$ is an ideal and the restriction of $g^{(1)}$ to this subalgebra vanishes. Actually, whether or not $\kappa_a = 0$, we always have:

$$g^{(1)} \Big|_{\mathfrak{f}_a^{(1)}} = \kappa_{1a} g_a^{(1)} \tag{3.11}$$

where $g_a^{(1)}$ is defined in the subspace $\mathfrak{f}_a^{(1)}$ as:

$$g_a^{(1)}(P_i, P_j) = \delta_{ij}\kappa_{ai} \qquad i, j = a, \ldots, N \tag{3.12}$$

and this suggests to take $g_a^{(1)}$ as the metric in the subspace $\mathfrak{f}_a^{(1)}$ of the tangent space $\mathfrak{p}^{(1)}$. When $\kappa_a \neq 0$ no real advantage is gained by considering $g_a^{(1)}$ further to $g^{(1)}$, because they are simply proportional, but when $\kappa_a = 0$, then the main metric vanishes when restricted to $\mathfrak{f}_a^{(1)}$ and the introduction of a *new* metric in this subspace is meaningful.

These special properties of the subspace $\mathfrak{f}_a^{(1)}$ of the tangent space at the origin when $\kappa_a = 0$ correspond to the existence of an invariant foliation of the space $\mathcal{S}^{[\kappa_1]\kappa_2,\ldots,\kappa_{a-1},0,\kappa_{a+1}\ldots,\kappa_N}$ itself in this case. This follows also quite clearly from the equation of the sphere (3.5), which reduces when $\kappa_a = 0$ to an equation involving only the variables $x^0, x^1, \ldots, x^{a-1}$. Each foliation leaf is coordinatised by the remaining variables $x^a, x^{a+1}, \ldots, x^N$, so that the subspace $\mathfrak{f}_a^{(1)}$ is the tangent space at the origin to the foliation leaf through the origin.

When there are more than one constant κ_a equal to zero, we have two nested foliations, and the extension of the preceeding ideas to this case is clear.

So the picture emerging from this description is the following: the quadratic metric structure of the rank-one space $\mathcal{S}^{(1)}$ is encoded in a main metric $g^{(1)}$ *and* a set of subsidiary metrics, denoted $g_a^{(1)}$, one for each zero constant $\kappa_a = 0$ in the

list $\kappa_2, \ldots, \kappa_N$. These subsidiary metrics are defined in each leaf of the invariant foliation(s) associated to the zero value of the constant(s) $\kappa_a = 0$.

In particular, when $\kappa_2 = 0$, the space $\mathcal{S}^{[\kappa_1]0,\kappa_3,\ldots,\kappa_N}$ has an invariant foliation whose set of leaves is $(x^0)^2 + \kappa_1 (x^1)^2 = 1 \equiv \mathcal{S}^{[\kappa_1]}$. Each leaf is described by the set of all the values of $x^2, \ldots, x^N$, and hence can be identified with a CK space $\mathcal{S}^{[0]\kappa_3,\ldots,\kappa_N}$. The main metric $g^{(1)}$ is degenerate and vanishes in each leaf. The subsidiary metric $g_2^{(1)}$ is well defined in each leaf. Should this look a bit involved, the paradigmatic example of this case is the Galilean spacetime, discussed in the last Section (cf. Table I below).

There is also an 'extrinsic' way to compute the main metric in $\mathcal{S}^{(1)}$: start from the CK group action in the ambient space $\mathbb{R}^{N+1}$ as a group of isometries of the *flat* metric:

$$(ds^2)_0^{(1)} = (dx^0)^2 + \sum_{l=1}^{N} \kappa_{0l} (dx^l)^2 \tag{3.13}$$

whose relation with the Killing–Cartan form is clear. As the space $\mathcal{S}^{(1)}$ is identified with the sphere (3.5), it would be obvious to consider the restriction of this flat metric to the sphere. This restriction turns out to be proportional to κ_1, and hence it vanishes when $\kappa_1 = 0$. It is therefore natural to consider the restriction of (3.13) to the sphere taking out the factor κ_1; this gives a well-defined non-trivial metric in all cases, no matter on whether κ_1 is zero or not, and this metric coincides with the one derived earlier by group theoretical reasoning.

Once the main metric is known, it is a simple matter to translate it to other coordinates. We give such an expression for two different set of coordinates. First, we define *Beltrami coordinates* for the CK space $\mathcal{S}^{(1)}$ as:

$$\eta^i := \frac{x^i}{x^0} \qquad i = 1, \ldots, N. \tag{3.14}$$

These coordinates (like Weierstrass ones) owe their name to the linear model of hyperbolic space, where they were first introduced. The main metric is given by:

$$(ds^2)^{(1)} = \frac{(1 + \kappa_1 \|\eta\|_\kappa^2)\, \|d\eta\|_\kappa^2 - \kappa_1 \langle \eta | d\eta \rangle_\kappa^2}{(1 + \kappa_1 \|\eta\|_\kappa^2)^2} \tag{3.15}$$

with $\eta = (\eta^1, \ldots, \eta^N)$, $d\eta = (d\eta^1, \ldots, d\eta^N)$, and where we have introduced two shorthands:

$$\langle a | b \rangle_\kappa := a^1 b^1 + \sum_{i=2}^{N} \kappa_{1i} a^i b^i \qquad \|a\|_\kappa^2 := \langle a | a \rangle_\kappa. \tag{3.16}$$

Further to Beltrami coordinates, the next natural choice is the geodesic parallel system of coordinates (3.6). Here the metric reads

$$(ds^2)^{(1)} = \prod_{l=2}^{N} \mathrm{C}^2_{\kappa_{0l}}(a^l)\, (da^1)^2 + \sum_{i=2}^{N-1} \kappa_{1i} \prod_{l=i+1}^{N} \mathrm{C}^2_{\kappa_{0l}}(a^l)\, (da^i)^2 + \kappa_{1N}\, (da^N)^2. \tag{3.17}$$

When $\kappa_1 = 0$ but all other $\kappa_i \neq 0$ this reduces to the flat space metric with suitable signature, $(ds^2)^{(1)} = (da^1)^2 + \sum_{i=2}^{N} \kappa_{1i} (da^i)^2$; in this case the space can be identified with $\mathbb{R}^N$ and a^i are cartesian coordinates.

4 Rank-two spaces

We focus now on the CK spaces of the $\mathcal{S}^{(2)}$ type, which are got by taking the quotient of $SO_{\omega_1,\dots,\omega_N}(N+1)$ by the subgroup $H^{(2)}$:

$$\mathcal{S}^{(2)} \equiv SO_{\omega_1,\dots,\omega_N}(N+1)/(SO_{\omega_1}(2)\otimes SO_{\omega_3,\dots,\omega_N}(N-1)). \tag{4.1}$$

The dimension of this space $\mathcal{S}^{(2)}$ is $2(N-1)$, and its rank is $\min(2, N-1)$, which is equal to 2 except in the special case $N=2$, where the rank is 1. In this Section we will understand we are working in the generic case $N>2$, so $\mathcal{S}^{(2)}$ will be actually a rank-two space. It is symmetric since the subgroup $H^{(2)}$ is generated by the subalgebra $\mathfrak{h}^{(2)}$ of the Cartan decomposition provided by the automorphism $\Theta^{(2)}$: $\mathfrak{so}_{\omega_1,\dots,\omega_N}(N+1) = \mathfrak{p}^{(2)} \oplus \mathfrak{h}^{(2)}$ where

$$\mathfrak{p}^{(2)} = \langle \Omega_{0j},\ \Omega_{1j} \quad j=2,\dots,N\rangle, \quad \mathfrak{h}^{(2)} = \langle \Omega_{01}; \quad \Omega_{ij}\ \ i,j=2,\dots,N\rangle. \tag{4.2}$$

Hence we have in $\mathcal{S}^{(2)}$ two sets of $(N-1)$ translations and two types of rotations. The structure of $\mathcal{S}^{(2)}$ can be more clearly appreciated by naming the abstract generators Ω_{ab} in the 'rank-adapted' notation as follows:

$$\begin{array}{c|cccc} \Omega_{01} & \Omega_{02} & \Omega_{03} & \dots & \Omega_{0N} \\ & \Omega_{12} & \Omega_{13} & \dots & \Omega_{1N} \\ \hline & & \Omega_{23} & \dots & \Omega_{2N} \\ & & & \ddots & \vdots \\ & & & & \Omega_{N-1\,N} \end{array} \quad \equiv \quad \begin{array}{c|cccc} -J_{(1)(2)} & P_{(2)1} & P_{(2)2} & \dots & P_{(2)N-1} \\ & P_{(1)1} & P_{(1)2} & \dots & P_{(1)N-1} \\ \hline & & J_{12} & \dots & J_{1\,N-1} \\ & & & \ddots & \vdots \\ & & & & J_{N-2\,N-1} \end{array}$$

Therefore we have introduced two sets of indices (a) and i with ranges $a=1,2$, and $i=1,\dots,N-1$. We complete the 'rank-two' notation by denoting the ω_i coefficients as:

$$\omega_1, \omega_2, \omega_3, \dots, \omega_N \quad \equiv \quad \kappa_{(2)}, \kappa_1, \kappa_2, \dots, \kappa_{N-1} \tag{4.3}$$

The value of these notational changes is clear once we write the commutation rules (2.1) which now read:

$$\begin{array}{lll} [J_{ij}, J_{ik}] = \kappa_{ij} J_{jk} & [J_{ij}, J_{jk}] = -J_{ik} & [J_{ik}, J_{jk}] = \kappa_{jk} J_{ij} \\ [J_{ij}, P_{(a)i}] = P_{(a)j} & [J_{ij}, P_{(a)j}] = -\kappa_{ij} P_{(a)i} & \\ [J_{(1)(2)}, P_{(1)i}] = P_{(2)i} & [J_{(1)(2)}, P_{(2)i}] = -\kappa_{(2)} P_{(1)i} & \\ [P_{(1)i}, P_{(1)j}] = \kappa_1 \kappa_{1i} J_{ij} & [P_{(2)i}, P_{(2)j}] = \kappa_1 \kappa_{1i} \kappa_{(2)} J_{ij} & \\ [P_{(1)i}, P_{(2)i}] = \kappa_1 \kappa_{1i} \kappa_{(2)} J_{(1)(2)}. & & \end{array} \tag{4.4}$$

Here again any two-index coefficient with two equal indices (as κ_{11}) will be always assumed as equal to 1. Note that the constant κ_1 (the old ω_2) is now appearing in all the commutators of the rank-two translations $P_{(a)i}$, foreshadowing again its role as the curvature of the space $\mathcal{S}^{(2)}$.

An important step in the rank-one case is the introduction of an ambient space on which the CK group acts linearly, and where $\mathcal{S}^{(1)}$ is embedded (the same idea can be succesfully applied for all CK spaces of any rank). All we need to carry out this idea is to replace the vector representation of the CK group (2.4) by another

representation, acting linearly on some space and having the subgroup $H^{(2)}$ as the isotropy subgroup. This is accomplished by taking the antisymmetrized square of the vector representation, which for the brevity sake will be called here the *bivector* representation of the CK group. Consider first the following $\frac{1}{2}N(N+1)\times\frac{1}{2}N(N+1)$ matrices where the matrix indices are pairs ij $(i<j)$ of indices in the set $0,1,\ldots,N$:

$$e_{ij,kl} \qquad i<j \qquad k<l \qquad i,j,k,l=0,1,\ldots,N \tag{4.5}$$

with an entry 1 in the row ij and column kl, with 0's otherwise. The explicit form of the bivector representation of the CK algebra, distinguised by a $^{(2)}$ superscript is:

$$\begin{aligned}
J_{ij}^{(2)} &= -\kappa_{ij}\sum_{s=j+2}^{N} e_{i+1s,j+1s} + \sum_{s=j+2}^{N} e_{j+1s,i+1s} - \kappa_{ij}\sum_{s=0}^{i} e_{si+1,sj+1} + \sum_{s=0}^{i} e_{sj+1,si+1}\\
&\quad +\kappa_{ij}\sum_{s=i+2}^{j} e_{i+1s,sj+1} - \sum_{s=i+2}^{j} e_{sj+1,i+1s}\\
J_{(1)(2)}^{(2)} &= \kappa_{(2)}\sum_{s=2}^{N} e_{0s,1s} - \sum_{s=2}^{N} e_{1s,0s}\\
P_{(1)j}^{(2)} &= -\kappa_{0j}\sum_{s=j+2}^{N} e_{1s,j+1s} + \sum_{s=j+2}^{N} e_{j+1s,1s} - \kappa_{0j}\, e_{01,0j+1} + e_{0j+1,01}\\
&\quad +\kappa_{0j}\sum_{s=2}^{j} e_{1s,sj+1} - \sum_{s=2}^{j} e_{sj+1,1s}\\
P_{(2)j}^{(2)} &= -\kappa_{(2)}\kappa_{0j}\sum_{s=j+2}^{N} e_{0s,j+1s} + \sum_{s=j+2}^{N} e_{j+1s,0s} + \kappa_{(2)}\kappa_{0j}\sum_{s=1}^{j} e_{0s,sj+1} - \sum_{s=1}^{j} e_{sj+1,0s}.
\end{aligned} \tag{4.6}$$

In the previous Section, the vector representation of the CK algebra generated a group of linear isometries in the ambient space $\mathbb{R}^{N+1}$, relative to the metric matrix $\Lambda_0^{(1)}$ (3.4) and whose isotropy subgroup was $H^{(1)}$. For the bivector representation we have similar properties, as well as a new one. First, each rank-two generator X (4.6) satisfies the condition $X^T\Lambda_0^{(2)}+\Lambda_0^{(2)}X$, where $\Lambda_0^{(2)}$ is the $\frac{1}{2}N(N+1)\times\frac{1}{2}N(N+1)$ matrix

$$\begin{aligned}
\Lambda_0^{(2)} &= e_{01,01} + \sum_{i=1}^{N-1}\kappa_{0i}\, e_{0i+1,0i+1} + \kappa_{(2)}\sum_{i=1}^{N-1}\kappa_{0i}\, e_{1i+1,1i+1}\\
&\quad +\kappa_{(2)}\sum_{i,j=1;i<j}^{N-1}\kappa_{0i}\kappa_{0j}\, e_{i+1j+1,i+1j+1}.
\end{aligned} \tag{4.7}$$

By direct exponentiation of the generators (4.6) we get a group of $\frac{1}{2}N(N+1)\times\frac{1}{2}N(N+1)$ matrices, the bivector representation of $SO_{\omega_1,\ldots,\omega_N}(N+1)$. This group acts linearly (by matrix multiplication) in the auxiliar 'bivector' ambient space $\mathbb{R}^{N(N+1)/2}=(x^{ij})$ $(i<j,\ i,j=0,\ldots N)$, as isometries of the metric (4.7), and therefore the action is not transitive. It is also clear that the subgroup $H^{(2)}$ is the isotropy subgroup of the point O with $x^{01}=1$ and all other coordinates $x^{ij}=0$ which will be taken as the origin. As a first step to the determination of orbits, we

first consider the sphere in $\mathbb{R}^{N(N+1)/2}$ corresponding to the metric (4.7):

$$(x^{01})^2+\sum_{i=1}^{N-1}\kappa_{0i}\,(x^{0i+1})^2+\kappa_{(2)}\sum_{i=1}^{N-1}\kappa_{0i}\,(x^{1i+1})^2+\kappa_{(2)}\sum_{i,j=1;i<j}^{N-1}\kappa_{0i}\kappa_{0j}\,(x^{i+1j+1})^2=1. \quad (4.8)$$

The coordinates x^{ij} in the ambient space are called *Plücker coordinates* for the space $\mathcal{S}^{(2)}$; they are the rank-two version of Weierstrass coordinates. All this seems quite analogous to the rank-one case. However there is an essential difference. The space $\mathcal{S}^{(2)}$ is to be identified with the orbit of O under the group action; this orbit, with dimension $2(N-1)$ cannot fill the sphere, of dimension $N(N+1)/2-1$. This is due to a new fact for the bivector representation of the CK group, which leaves invariant not only the quadratic form (4.7) but also a set of quadratic relations known as *Plücker relations* (also Grassmann relations or even p-relations):

$$x^{ij}x^{kl}-x^{ik}x^{jl}+x^{il}x^{jk}=0 \qquad i<j<k<l \qquad i,j,k,l=0,\ldots,N \qquad (4.9)$$

so actually the rank-two CK space $\mathcal{S}^{(2)}$ should be identified to the intersection of the sphere (4.8) with the family of quadratic cones (4.9) in the bivector ambient space. We can profit from this new fact: in some open neighbourhood of the origin of $\mathcal{S}^{(2)}$, we can take x^{0j} and x^{1j} ($j=2,\ldots,N$) as the $2(N-1)$ independent coordinates of the space. The coordinate x^{01} will be left as a non-independent one, and the remaining will be eliminated by using the Plücker equations with indices $01kl$:

$$x^{kl}=\frac{x^{0k}x^{1l}-x^{0l}x^{1k}}{x^{01}} \qquad k,l=2,\ldots,N. \qquad (4.10)$$

With this choice for the inessential coordinates $x^{kl},(k,l=2,\ldots,N)$ it can be indeed shown that *all* the Plücker equations (and not only those with indices $01kl$) become identities. After this is done, everything is similar to the rank-one case: we have some essential coordinates $x^{0i},x^{1i},\ i=2,\ldots,N-1$ and a single auxiliar inessential coordinate x^{01} which can be eliminated by the equation of the sphere after (4.10) has been used.

Once we have got an explicit description of the space $\mathcal{S}^{(2)}$, we turn to its quadratic metric. Similarly to the rank-one case, there is a main and an eventual set of subsidiary metrics in the space $\mathcal{S}^{(2)}$, all of which are required in order to give a complete description of its metric structure. The main metric $g^{(2)}$ comes again from the Killing–Cartan form on the algebra (2.12), when suitably restricted to the tangent space $\mathfrak{p}^{(2)}$ to the origin of $\mathcal{S}^{(2)}$. We first give the expressions for the diagonal non-identically zero values of the Killing–Cartan form

$$\begin{array}{ll} g(P_{(1)i},P_{(1)i})=-2(N-1)\kappa_1\kappa_{1i} & g(P_{(2)i},P_{(2)i})=-2(N-1)\kappa_{(2)}\kappa_1\kappa_{1i} \\ g(J_{(1)(2)},J_{(1)(2)})=-2(N-1)\kappa_{(2)} & g(J_{jk},J_{jk})=-2(N-1)\kappa_{jk} \end{array} \qquad (4.11)$$

where the constant κ_1 appears again only in the restriction to the $\mathfrak{p}^{(2)}$ subspace. It is clear that the natural candidate for the metric in the tangent space $\mathfrak{p}^{(2)}$ is obtained by writing the restriction of the Killing–Cartan in the Lie algebra to the subspace $\mathfrak{p}^{(2)}$ as:

$$g\,|_{\mathfrak{p}^{(2)}}=-2(N-1)\kappa_1 g^{(2)} \qquad (4.12)$$

that is:

$$g^{(2)}(P_{(1)i},P_{(1)i})=\kappa_{1i} \qquad g^{(2)}(P_{(2)i},P_{(2)i})=\kappa_{(2)}\kappa_{1i}. \tag{4.13}$$

At the origin of the space $\mathcal{S}^{(2)}$, and in the basis of the tangent space provided by the translation generators $P_{(a)i}$ themselves, the main metric $g^{(2)}$ is given by the matrix:

$$\Lambda^{(2)}=\begin{pmatrix}\Pi & 0\\ 0 & \kappa_{(2)}\Pi\end{pmatrix} \qquad \Pi=\ \mathrm{diag}\,(+,\,\kappa_{12},\,\kappa_{13},\,\ldots,\,\kappa_{1\,N-1}) \tag{4.14}$$

so the signature is determined by the constants $\kappa_{(2)},\kappa_2,\ldots,\kappa_{N-1}$. This metric can be translated to all points in the CK space $\mathcal{S}^{(2)}$ by the group action, so that the CK group acts by isometries in $\mathcal{S}^{(2)}$. Now the sectional curvatures of this metric can be computed, and the result can be foreseen from the commutation relations (4.4): sectional curvatures are not constant, but they are as close to constant as a rank-two space would allow. At the origin, the sectional curvature of the space $\mathcal{S}^{(2)}$ along any 2-plane direction spanned by any two tangent vectors $P_{(a)i},P_{(a)j}$ or $P_{(a)i},P_{(b)i}$ is constant and equal to κ_1 (two vectors where both indices are different span a 2-plane for which the curvature is always identically equal to zero, no matter of the values of the constant κ_1; this is behind the classical definition of the rank of the homogeneous space associated to a simple group). This completes the geometric interpretation of the constants κ_i in the geometry of the space $\mathcal{S}^{(2)}$, and suggest a complete notation for these rank-two spaces: $\mathcal{S}^{(2)}\equiv\mathcal{S}^{\kappa_{(2)}[\kappa_1]\kappa_2\ldots\kappa_{N-1}}$.

From this point onwards things are rather similar to the rank-one case. Invariant foliations appear when any κ in the set $\kappa_{(2)},\kappa_2,\kappa_3,\ldots,\kappa_{N-1}$ is equal to zero (when the main metric is degenerate). We start by introducing a decomposition of $\mathfrak{p}^{(2)}$ as:

$$\begin{array}{ll}\mathfrak{p}^{(2)}=\mathfrak{b}_a{}^{(2)}\oplus\mathfrak{f}_a^{(2)} & a=(2),2,\ldots N-1\\ \mathfrak{b}_{(2)}^{(2)}=\langle P_{(1)1},\ldots,P_{(1)N-1}\rangle & \mathfrak{b}_a^{(2)}=\langle P_{(1)1},\ldots,P_{(1)a-1}\,;\,P_{(2)1},\ldots,P_{(2)a-1}\rangle\\ \mathfrak{f}_{(2)}^{(2)}=\langle P_{(2)1},\ldots,P_{(2)N-1}\rangle & \mathfrak{f}_a^{(2)}=\langle P_{(1)a},\ldots,P_{(1)N-1}\,;\,P_{(2)a},\ldots,P_{(2)N-1}\rangle.\end{array} \tag{4.15}$$

When $\kappa_{(2)}=0$, (resp. $\kappa_a=0$) then $\mathfrak{f}_{(2)}^{(2)}$ (resp. $\mathfrak{f}_a^{(2)}$) is an ideal and the restriction of $g^{(2)}$ to this subalgebra vanishes. Whether or not $\kappa_{(2)}=0$ or $\kappa_a=0$, we always have:

$$g^{(2)}\Big|_{\mathfrak{f}_{(2)}^{(2)}}=\kappa_{(2)}g_{(2)}^{(2)} \qquad g^{(2)}\Big|_{\mathfrak{f}_a^{(2)}}=\kappa_{1a}g_a^{(2)} \qquad a=2,\ldots,N-1 \tag{4.16}$$

where $g_{(2)}^{(2)}$ is defined in $\mathfrak{f}_{(2)}^{(2)}$ (resp. $g_a^{(2)}$ is defined in $\mathfrak{f}_a^{(2)}$) as:

$$\begin{array}{l}g_{(2)}^{(2)}(P_{(2)i},P_{(2)j})=\delta_{ij}\kappa_{1i} \qquad i,j=1,\ldots,N-1\\ g_a^{(2)}(P_{(1)i},P_{(1)j})=\delta_{ij}\kappa_{ai} \quad g_a^{(2)}(P_{(2)i},P_{(2)j})=\delta_{ij}\kappa_{(2)}\kappa_{ai} \quad i,j=a,\ldots,N-1.\end{array} \tag{4.17}$$

Even when $\kappa_{(2)}=0$ or $\kappa_a=0$, these define metrics in the subspaces $\mathfrak{f}_{(2)}^{(2)}$ or $\mathfrak{f}_a^{(2)}$. In this case the complete metric description of the space splits into a degenerate main metric, and a metric in each of the fibers. When a constant $\kappa_{(2)},\kappa_2,\ldots,\kappa_{N-1}$ is zero, the ad-invariance of the corresponding subalgebra $\mathfrak{f}_{(2)}^{(2)}$ or $\mathfrak{f}_a^{(2)}$ produce invariant foliations in the rank-two space. For instance, when $\kappa_{(2)}=0$, the equation (4.8) is:

$$(x^{01})^2+\kappa_{01}\,(x^{02})^2+\ldots+\kappa_{0N-1}\,(x^{0N})^2=1 \tag{4.18}$$

so the set of leaves (the base space for the fibered structure) has dimension $N-1$ and can be identified with a rank-one space $\mathcal{S}^{[\kappa_1]\kappa_2,\ldots,\kappa_{N-1}}$. Each leaf in the foliation (the fiber), characterized by some fixed values of the coordinates $x^{02},\ldots,x^{0N}$, is described by the remaining essential coordinates $x^{12},x^{13},\ldots,x^{1N}$; the fiber can be identified with the rank-one space $\mathcal{S}^{[0]\kappa_2,\ldots,\kappa_{N-1}}$.

When $\kappa_2=0$, the equation (4.8) is:

$$(x^{01})^2+\kappa_1\,(x^{02})^2+\kappa_{(2)}\kappa_1\,(x^{12})^2=1 \tag{4.19}$$

so the set of leaves has now dimension 2, and can be identified with the two-rank space $\mathcal{S}^{\kappa_{(2)}[\kappa_1]}$. A set of coordinates for each leaf is specified by the remaining $x^{03},x^{04},\ldots,x^{0N}$ and $x^{13},x^{14},\ldots,x^{1N}$. As a CK space the fiber is the rank-two space is $\mathcal{S}^{\kappa_{(2)}[\kappa_1]\kappa_3,\ldots,\kappa_{N-1}}$. The situation is similar when $\kappa_3=0,\ldots$, etc.

An alternative approach to compute these metrics is to start from the flat metric in the bivector ambient space, then restrict it to the intersection of the sphere (4.8) with the Plücker cones and take out the coefficient of κ_1 in the expression thus obtained. The flat bivector ambient metric is:

$$\begin{aligned}(ds)_0^{(2)}=(dx^{01})^2+\kappa_1\sum_{i=1}^{N-1}\kappa_{1i}\,(dx^{0\,i+1})^2+\kappa_1\kappa_{(2)}\sum_{i=1}^{N-1}\kappa_{1i}\,(dx^{1\,i+1})^2\\ +\kappa_1\kappa_{(2)}\sum_{i,j=1;i<j}^{N-1}\kappa_{1i}\kappa_{0j}\,(dx^{i+1\,j+1})^2.\end{aligned} \tag{4.20}$$

Working locally in some open neighbourhood of the origin in $\mathcal{S}^{(2)}$ (determined by the condition $x^{01}>0$), we can expressed (4.20) in terms of x^{01},x^{0i},x^{1i} by means of (4.10); then we restrict to the sphere. We carry out this programm for the so-called *Beltrami coordinates* in the rank-two space defined by

$$\eta^i:=\frac{x^{0\,i+1}}{x^{01}}\qquad \xi^i:=\frac{x^{1\,i+1}}{x^{01}}\qquad i=1,\ldots,N-1. \tag{4.21}$$

The equation of the sphere (4.8) turns into

$$(x^{01})^2\,(1+\kappa_1\|(\eta,\xi)\|_\kappa^2)=1 \tag{4.22}$$

where we introduce a notational shorthand, analogous to those introduced in the rank-one case: for $\eta=(\eta^1,\ldots,\eta^{N-1})$, $\xi=(\xi^1,\ldots,\xi^{N-1})$, $\|(.,.)\|_\kappa$ is defined by

$$\|(\eta,\xi)\|_\kappa^2:=\sum_{i=1}^{N-1}\kappa_{1i}\,(\eta^i)^2+\kappa_{(2)}\sum_{i=1}^{N-1}\kappa_{1i}\,(\xi^i)^2+\kappa_{(2)}\sum_{i,j=1;i<j}^{N-1}\kappa_{1i}\kappa_{0j}\,(\eta^i\xi^j-\eta^j\xi^i)^2. \tag{4.23}$$

We remark that the expression $\|(\eta,\xi)\|_\kappa$ is only a norm in the standard sense of the term (i.e., definite positive) when all κ constants involved $\kappa_{(2)},\kappa_2,\ldots,\kappa_{N-1}$ are positive. Otherwise it is either indefinite or degenerate. We also introduce the companion notational shorthand:

$$\langle(\eta,\xi)|(d\eta,d\xi)\rangle_\kappa:=\sum_{i=1}^{N-1}\kappa_{1i}\,\eta^i\,d\eta^i+\kappa_{(2)}\sum_{i=1}^{N-1}\kappa_{1i}\,\xi^i\,d\xi^i$$

$$+\kappa_{(2)}\sum_{i,j=1;i<j}^{N-1}\kappa_{1i}\kappa_{0j}\,(\eta^i\xi^j-\eta^j\xi^i)\,d(\eta^i\xi^j-\eta^j\xi^i). \quad (4.24)$$

with $d\eta=(d\eta^1,\ldots,d\eta^{N-1})$ and $d\xi=(d\xi^1,\ldots,d\xi^{N-1})$, as well as

$$\|(d\eta,d\xi)\|^2_\kappa:=\sum_{i=1}^{N-1}\kappa_{1i}(d\eta^i)^2+\kappa_{(2)}\sum_{i=1}^{N-1}\kappa_{1i}(d\xi^i)^2+\kappa_{(2)}\sum_{i,j=1;i<j}^{N-1}\kappa_{1i}\kappa_{0j}(d(\eta^i\xi^j-\eta^j\xi^i))^2. \quad (4.25)$$

Then we find that the flat ambient metric, when restricted to $\mathcal{S}^{(2)}$ is proportional to κ_1; by taking out this factor the end result for the main metric (4.20) in the space $\mathcal{S}^{(2)}$ is expressed as

$$(ds^2)^{(2)}=\frac{(1+\kappa_1\|(\eta,\xi)\|^2_\kappa)\,\|(d\eta,d\xi)\|^2_\kappa-\kappa_1\,\langle(\eta,\xi)|(d\eta,d\xi)\rangle^2_\kappa}{(1+\kappa_1\|(\eta,\xi)\|^2_\kappa)^2}. \quad (4.26)$$

Notice the close ressemblance of this metric with its rank-one analogous (3.15).

A last comment is in order. In general, the Plücker relations $x^{ij}x^{kl}-x^{ik}x^{jl}+x^{il}x^{jk}=0$ are invariant under the bivector representation, but the r.h.s. of the relation is not. An exception is the lowest dimensional rank-two case $N=3$. Here there is a single Plücker relation 0123, and the quadratic form $x^{01}x^{23}-x^{02}x^{13}+x^{03}x^{12}$ is also invariant; this is related to the (exceptional) known quadratic Riemannian metric in the real Grassmannian of two-planes in four dimensions. In any other real Grassmannian, there is up to a factor a unique Riemannian quadratic metric, which should coincide with the one we have derived.

5 Curvature and metric in spacetimes and phase spaces

We now turn to the physical interpretation for some of the rank-two spaces we have studied here: just as the rank-one CK spaces afford models for all homogeneous spacetimes (e.g., those in the Bacry and Levy–Leblond classification [6]), their corresponding rank-two spaces gives models for the phase spaces of a free system whose spacetime is a rank-one CK space. We will insist on the metric aspect, as usually the metric structure of the phase space is disregarded in favour of the symplectic structure, upon which we have said nothing, and the metric structure which naturally appear in our scheme cannot be easily guessed from what is more or less implicit in the literature when dealing with phase spaces for curved spacetimes. We restrict here to pointing out the most relevant traits.

First, a rather elementary remark is that the grouping of the $2(N-1)$ coordinates of a rank-two space into two sets is intimately related to the existence of pairs of canonically conjugated variables: the *momentum*-like η^i and *position*-like ξ^i coordinates. Second, a warning: the notation we have tailored for rank-two spaces tries to convey the meaning of quantities in the most close form as possible to the rank-one case. We have used in both cases the name κ_1 for the curvature of the space. This

is satisfactory as long as we deal with either rank-one *or* rank-two spaces alone, but turns rather confusing when we insist on considering simultaneously associated rank-one and rank-two spaces. In this case it is far better to go back to the 'neutral' ω notation used in the Section 2 of the paper. Each constant ω_i will be interpreted differently according as we are working in $\mathcal{S}^{(1)}$ or $\mathcal{S}^{(2)}$; the complete notation for these spaces, $\mathcal{S}^{(1)} \equiv \mathcal{S}^{[\omega_1]\omega_2,\omega_3,\ldots,\omega_N}$ or $\mathcal{S}^{(2)} \equiv \mathcal{S}^{\omega_1[\omega_2]\omega_3,\ldots,\omega_N}$ is informative enough to clear any misunderstanding.

We display in Table I nine especially relevant CK spaces $\mathcal{S}^{(1)}$ in the general N-dimensional case. These are the spaces $\mathcal{S}^{[\omega_1]\omega_2,+,\ldots,+}$ associated to the algebras $so_{\omega_1,\omega_2,+,\ldots,+}(N+1)$ where the constants $\omega_3,\ldots,\omega_N$ are all positive. The first row gives the usual Riemannian spaces with a non-degenerate and positive definite metric. The six remaining spaces are the six well known possible homogeneous kinematical spacetimes in $(N-1)+1$ dimensions. In the second row we find the 'absolute-time' models associated to Newtonian spacetimes with the three possible values of the spacetime curvature (oscillating Newton–Hooke (NH), with positive spacetime curvature and group $T_{2N-2}(SO(N-1)\otimes SO(2))$; Galilei, with zero curvature; and expanding NH, with negative curvature and group $T_{2N-2}(SO(N-1)\otimes SO(1,1))$). These three spaces have a main degenerate metric (whose length is the absolute time) and a subsidiary well defined metric (the purely spatial metric, which only makes sense when taken on each of the leaves of the invariant foliation, here the leaves of absolute simultaneity). In the third row the 'relative-time' spacetime models corresponding to relativistic spacetimes with a Lorentz type metric appear.

Table I. The N-dimensional rank-one spaces $\mathcal{S}^{[\omega_1]\omega_2,+,\ldots,+}$.

Riemannian spaces $\mathcal{S}^{[\omega_1]+,\ldots,+}$: $\Lambda^{(1)} = \mathrm{diag}\,(+,+,\ldots,+)$ No invariant foliation		
Elliptic Space	Euclidean Space	Hyperbolic Space
$\mathcal{S}^{[+]+,\ldots,+} \simeq \mathbf{S}^N$	$\mathcal{S}^{[0]+,\ldots,+} \simeq \mathbf{E}^N$	$\mathcal{S}^{[-]+,\ldots,+} \simeq \mathbf{H}^N$
$SO(N+1)/SO(N)$	$ISO(N)/SO(N)$	$SO(N,1)/SO(N)$
Positive Curvature	Zero Curvature	Negative Curvature
No Invariant Foliation	No Invariant Foliation	No Invariant Foliation
SemiRiemannian spaces $\mathcal{S}^{[\omega_1]0,+,\ldots,+}$: $\Lambda^{(1)} = \mathrm{diag}\,(+,0,\ldots,0)$ $\Lambda_2^{(1)} = \mathrm{diag}\,(+,\ldots,+)$ Invariant foliation: Set of foliation leaves $\mathcal{S}^{[\omega_1]}$ Fiber $\mathcal{S}^{[0],+,\ldots,+}$		
Oscillating NH Spacetime	Galilean Spacetime	Expanding NH Spacetime
$\mathcal{S}^{[+]0,+,\ldots,+}$	$\mathcal{S}^{[0]0,+,\ldots,+}$	$\mathcal{S}^{[-]0,+,\ldots,+}$
$ONH/ISO(N-1)$	$IISO(N-1)/ISO(N-1)$	$ENH/ISO(N-1)$
Positive Curvature	Zero Curvature	Negative Curvature
Invariant Foliation	Invariant Foliation	Invariant Foliation
Base $\mathcal{S}^{[+]}$, Fiber $\mathcal{S}^{[0],+,\ldots,+}$	Base $\mathcal{S}^{[0]}$, Fiber $\mathcal{S}^{[0],+,\ldots,+}$	Base $\mathcal{S}^{[-]}$, Fiber $\mathcal{S}^{[0],+,\ldots,+}$
PseudoRiemannian spaces $\mathcal{S}^{[\omega_1]-,+,\ldots,+}$: $\Lambda^{(1)} = \mathrm{diag}\,(+,-,\ldots,-)$ No invariant foliation		
Anti-DeSitter Spacetime	Minkowskian Spacetime	DeSitter Spacetime
$\mathcal{S}^{[+]-,+,\ldots,+}$	$\mathcal{S}^{[0]-,+,\ldots,+}$	$\mathcal{S}^{[-]-,+,\ldots,+}$
$SO(N-1,2)/SO(N-1,1)$	$ISO(N-1,1)/SO(N-1,1)$	$SO(N,1)/SO(N-1,1)$
Positive Curvature	Zero Curvature	Negative Curvature
No Invariant Foliation	No Invariant Foliation	No Invariant Foliation

In terms of the rank-one spaces, $\mathcal{S}^{[\omega_1]\omega_2,+,\ldots,+}$, ω_1 is the curvature and ω_2 determines the signature of the main metric. A non-zero positive ω_1 is related to the

oscillating NH or Anti-DeSitter radius as $\omega_1 = 1/R^2$ or to the expanding NH or DeSitter characteristic length when it is negative, $\omega_1 = -1/R^2$. The main metric is definite positive when $\omega_2 > 0$ –i.e., in the Euclidean, elliptic, and hyperbolic spaces–, degenerate when $\omega_2 = 0$ –i.e., in the Galilean and both NH spacetimes– and Lorentzian type when $\omega_2 < 0$ –i.e., in Minkowski and both DeSitter spacetimes–; the relation of ω_2 with the standard relativistic constant c is $\omega_2 = -1/c^2$.

In the kinematical spaces, and in parallel coordinates, a^1 should be identified with the time coordinate, the others being space coordinates. In particular, in the three $(1+3)$ Newtonian spacetimes ($N = 4, \kappa_2 = 0, \kappa_3 = 1, \kappa_4 = 1$), the main metric (3.17) reduces to $(ds^2)^{(1)} = (da^1)^2$, which gives a length which depends only on the end points of the curve, but not on the path itself. The leaves of the foliation are characterized by a^1 constant, and should be identified in the three cases with an Euclidean three-space, the subsdiary metric being given by $g_2^{(1)} = \frac{1}{\kappa_2} g^{(1)}|_{\mathfrak{f}_2^{(1)}}$:

$$(ds^2)_2^{(1)} = (da^2)^2 + (da^3)^2 + (da^4)^2 \qquad (a^1 \text{ constant}). \tag{5.27}$$

We now describe the metric structure of the phase spaces associated to the nine CK rank-one spaces $\mathcal{S}^{[\omega_1]\omega_2,+,\dots,+}$ of Table I whose corresponding rank-two spaces (identified with the space of lines in $\mathcal{S}^{[\omega_1]\omega_2,+\dots,+}$) are $\mathcal{S}^{\omega_1[\omega_2]+,\dots,+}$. Their geometric properties, as far as their metrics are concerned, depend also on the values of the constants ω_1, ω_2 but in a rather different way to which it was formerly the case. Here the constant ω_1 appears in the signature of the main metric, so that whenever $\omega_1 = 0$ the main metric is degenerate, (and we are in presence of an invariant foliation in the phase space). This happens for the three rank-two spaces $\mathcal{S}^{0[\omega_2]+,\dots,+}$ with $\omega_1 = 0$; depending on whether $\omega_2 > 0, \omega_2 = 0, \omega_2 = -1/c^2 < 0$, these spaces are the sets of lines in the Euclidean, Galilean and Minkowskian rank-one spaces. All these phase spaces $\mathcal{S}^{0[\omega_2]+,\dots,+}$ have an invariant foliation, with an $(N-1)$-dimensional base space and $(N-1)$-dimensional leaves. The base space is the rank-one space $\mathcal{S}^{[\omega_2]+,\dots,+}$ identified in kinematical terms with the *velocity* space, with momentum-like coordinates $\eta^1, \dots, \eta^{N-1}$. The degenerate main metric (4.26) reduces to a metric in the base $(N-1)$-velocity space. Each leaf in the foliation is coordinatised by the values of the position-like coordinates $\xi^1, \dots, \xi^{N-1}$. All these results should have been expected: when $\omega_1 = 0$ the rank-one spacetime has zero curvature; in this case parallelism of lines is absolute, and we can meaningfully class all lines into parallelism classes, each of which is completely described by its (common) velocity. In particular, when $\omega_1 = 0, \omega_3 = \omega_4 = 1$ and $N = 4$ the expression for the degenerate main metric in $\mathcal{S}^{0[\omega_2]+,+}$ is:

$$(ds)^{(2)} = \frac{(1 + \omega_2 \|\eta\|^2)\, \|d\eta\|^2 - \omega_2 \langle \eta | d\eta \rangle^2}{(1 + \omega_2 \|\eta\|^2)^2} \tag{5.28}$$

$$\begin{array}{ll} \|\eta\|^2 = (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 & \|d\eta\|^2 = (d\eta^1)^2 + (d\eta^2)^2 + (d\eta^3)^2 \\ \langle \eta | d\eta \rangle^2 = \eta^1\, d\eta^1 + \eta^2\, d\eta^2 + \eta^3\, d\eta^3 & \end{array} \tag{5.29}$$

which corresponds to the fact that in these cases three-velocity space is a rank-one space of constant curvature ω_2. The hyperbolic nature of the velocity space

in relativity has been known since a long time. For the Galilean case $\mathcal{S}^{0[0]+,+}$ the metric in phase space degenerates into a metric on three-velocity space, which is a flat Euclidean space, with a simpler expression:

$$(ds)^{(2)} = (d\eta^1)^2 + (d\eta^2)^2 + (d\eta^3)^2. \tag{5.30}$$

Notice that in both cases the interpretation is the same: distance in phase space comes exclusively from the distance in the base 'velocity space', where it reduces to the relative speed between two free movements. In these three $\omega_1 = 0$ cases there is an invariant foliation, their leaves being identified with the Euclidean 3d position space; the subsidiary metric $g_2^{(2)}$ defined in each leaf can be obtained from (4.26).

However, when $\omega_1 \neq 0$ (either in NH or in DeSitter spacetimes), the invariant foliation of the phase space disappears, and the possibility of using a 'reduced' three dimensional velocity space does no longer exists: phase space should be approached as the six-dimensional space it is.

The value of the second constant ω_2 determines another aspect of the geometrical nature of the phase space: its curvature. The three non-relativistic phase spaces (where $\omega_2 = 0$), i.e., oscillating NH $\mathcal{S}^{+[0]+,\dots,+}$, Galilei $\mathcal{S}^{0[0]+,\dots,+}$ and expanding NH $\mathcal{S}^{-[0]+,\dots,+}$ have *zero* curvature (compare with their spacetime version). This means that these phase spaces are six-dimensional *affine* spaces. However, the three relativistic phase spaces (anti-DeSitter $\mathcal{S}^{-[-1/c^2]+,\dots,+}$, Minkowskian $\mathcal{S}^{0[-1/c^2]+,\dots,+}$ and DeSitter $\mathcal{S}^{+[-1/c^2]+,\dots,+}$ have negative curvature; only the Minkowskian phase space has an invariant foliation with a three-dimensional base space of negative curvature. At this point we sum up and display the results in Table II, which is laid similarly to Table I to make the comparation easy.

Table II. The $2(N-1)$-dimensional rank-two spaces $\mathcal{S}^{\omega_1[\omega_2]+,\dots,+}$.

Elliptic line-space $\mathcal{S}^{+[+]+,\dots,+}$ $SO(N+1)/SO(2)\otimes SO(N-1)$ Positive Curvature No Invariant Foliation	Euclidean line-space $\mathcal{S}^{0[+]+,\dots,+}$ $ISO(N)/\mathbb{R}\otimes SO(N-1)$ Positive Curvature Invariant Foliation Base $\mathcal{S}^{[+]+,+}$, Fiber $\mathcal{S}^{[0]+,+}$	Hyperbolic line-space $\mathcal{S}^{-[+]+,\dots,+}$ $SO(N,1)/SO(1,1)\otimes SO(N-1)$ Positive Curvature No Invariant Foliation
Oscillating NH Phase Space $\mathcal{S}^{+[0]+,\dots,+}$ $ONH/SO(2)\otimes SO(N-1)$ Zero Curvature No Invariant Foliation	Galilean Phase Space $\mathcal{S}^{0[0]+,\dots,+}$ $IISO(N-1)/\mathbb{R}\otimes SO(N-1)$ Zero Curvature Invariant Foliation Base $\mathcal{S}^{[0]+,+}$, Fiber $\mathcal{S}^{[0]+,+}$	Expanding NH Phase Space $\mathcal{S}^{-[0]+,\dots,+}$ $ENH/SO(1,1)\otimes SO(N-1)$ Zero Curvature No Invariant Foliation
Anti-DeSitter Phase Space $\mathcal{S}^{+[-]+,\dots,+}$ $SO(N-1,2)/SO(2)\otimes SO(N-1)$ Negative Curvature No Invariant Foliation	Minkowskian Phase Space $\mathcal{S}^{0[-]+,\dots,+}$ $ISO(N-1,1)/\mathbb{R}\otimes SO(N-1)$ Negative Curvature Invariant Foliation Base $\mathcal{S}^{[-]+,+}$, Fiber $\mathcal{S}^{[0]+,+}$	DeSitter Phase Space $\mathcal{S}^{-[-]+,\dots,+}$ $SO(N,1)/SO(1,1)\otimes SO(N-1)$ Negative Curvature No Invariant Foliation

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